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# METHODS OF MATHEMATICAL SIMULATION AND MACHINE IDENTIFICATION OF ANOMALOUS DIFFUSION PROCESSES

***For the class of anomalous diffusion processes, the mathematical models of which are formalized in the form of variational inequalities in partial derivatives, a method of mathematical modeling based on the optimization procedure is proposed. The method is considered in relation to the generalized mathematical model of the studied class of anomalous diffusion processes. Which made it possible to ensure the principle of unification and typification in the application of this method, as well as the correctness of using the generalized mathematical model in applied problems of mathematical modeling of known industrial and practically important natural cases of anomalous diffusion processes. At the same time, the task of implementing mathematical models of anomalous diffusion processes based on the proposed method is reduced to finding the maximum of the Hamiltonian function defined in the state space of the processes under consideration. A method of parametric identification of mathematical models of anomalous diffusion processes in the formulation of the problem of optimal control is also proposed. The method is reduced to the use of the optimization procedure of the gradient projection method. The possibility of solving the problem of parametric identification in cases of both linear and non-linear mathematical models of anomalous diffusion processes is proved. Moreover, the nonlinear formulation of the parametric identification problem does not lead to computational implementation complications, since the solution is based only on finding the gradient projection of the state function of the anomalous diffusion process. The proposed methods are presented in strict compliance with the provisions of functional analysis, which ensures their correctness and adequacy in solving a wide range of applied problems.***

***Keywords: anomalous diffusion process, mathematical model, variation, variational inequality, optimization, principle of unification and typification, gradient, parametric identification.***

**Introduction.** In a number of important applied tasks technological (or naturally) processes are characterized by deviations from well-known physical laws. In this regard these processes received in special literature the name anomalous (in particular, abnormal diffusive) [1 — 4]. First of all, the reological processes connected with mining can be an example of such processes. For the description of abnormal diffusive processes, as the adequate mathematical models (MM) it was offered to use the device of variation inequalities in private derivatives [5 — 8].

As it was shown in work [9], in practical appendices it is most convenient to use the following formalization of abnormal diffusive processes.

Let the function $\psi(t,\bar{z})$, defined on a bounded open set $\Omega$ of the space $\Re^n$, $n=1,2$, with smooth boundary $\Gamma$ and the time interval $(0,t_k)$ for $t_k<\infty$, $Q=\Omega\times(0,t_k)$, $\Sigma=\Gamma\times(0,t_k)$ is the solution of the variational inequality

$$\psi\in K:\left(m(\bar{z})\frac{\partial\psi}{\partial t},v-\psi\right)+\left(B(\gamma)\psi,v-\psi\right)+j(v)-j(u)\geq$$

$$\geq\left(f,v-\psi\right)\quad\forall v\in H^1(\Omega),\qquad(1)$$

$$\psi(0,\bar{z})=\psi_0(\bar{z}),\qquad(2)$$

where the operator $B(\gamma)$ specifies a linear transformation $B(\gamma): H^1(\Omega)\to H^1(\Omega)$ and is defined by the bilinear form:

$$\left(B(\gamma)\psi, v-\psi\right)=\int_{\Omega}\left(\sum_{i=1}^{n}\frac{\partial\psi}{\partial z_i}\cdot\frac{\partial(v-\psi)}{\partial z_i}\right)d\bar{z}\,, \tag{3}$$

$f$ – the driving function of the process, for which the operation $\left(f, v-\psi\right)$ coincides with the scalar product in $L^2(\Omega)$, i.e.

$$\left(f, v-\psi\right)=\int_{\Omega}\left[f(\bar{z}), v-\psi\right]d\Omega \text{ or } \left(f, v-\psi\right)=\int_{\Gamma}\left[f(\bar{z}), v-\psi\right]d\Gamma$$

(hereinafter, for simplicity, restrict ourselves to the tasks at the border $\Gamma$); $j(\cdot)$ — convex functionals defining the kind of physical process in reology and which are specified as follows

$$j(\cdot)=\int_{\Gamma}\varphi(\psi,\bar{z})\cdot\lambda(\psi)\,d\Gamma\,,\quad j(\cdot)=\int_{\Omega}\varphi(\psi,\bar{z})\cdot\lambda(\psi)\,d\Omega\,. \tag{4}$$

In the relation (4) accept that $\varphi(\cdot)$ – is a continuous function, $\lambda(\cdot)$ — is continuous differentiable or not having the properties of differentiable functions.

Space of admissible functions $\varphi(\cdot)$ and $\lambda(\cdot)$ are defined as $\Delta\in L^{\infty}(\overline{Q}),\ \Lambda\in L^{\infty}(\overline{Q})$ where it is assumed that $\varphi(\cdot), \lambda(\cdot)\in L^{\infty}(\overline{Q})$, $\overline{Q}=\overline{\Omega}\times(0,t_k)$ and the spaces $\Delta$ and $\Lambda$ are Banach with respect to the norm

$$\left\|\varphi(\psi,\bar{z})\right\|_{\Delta}=\left\|\varphi(\psi,\bar{z})\right\|_{L^{\infty}(\overline{Q})}.$$

**The aim of the study.** The purpose of the research is to develop methods of numerical implementation (mathematical modeling) and parametric identification of mathematical models of anomalous diffusion processes, presented in the form of variational inequalities using the principle of unification and typification.

**Presentation of the main research material**. First, we will consider the method of computational implementation of mathematical models of anomalous diffusion processes. At the same time, based on the principle of unification and typification, in further considerations we will use the generalized MM of the studied processes.

**1. Metod of mathematical modelling of abnormal diffusive processes.** The proposed metod for solving variational inequalities of the form (1), (2) is based on the proof of the following statements.

To find the optimal solution $\psi(t,\bar{z})$ of the variational inequality (1), (2) there must exist a nonzero continuous function $p(t,\bar{z})$, so that at any time $t$ in the interval $0\le t\le T$ ($T$ — time of physical processes) the Hamiltonian function $\widetilde{H}$ in the spatial domain $\Omega$ (or on its boundary $\Gamma$) would take the maximum value, where

$$\widetilde{H}=\left\langle\left(\left(B(\gamma)\widetilde{\psi},\widetilde{v}-\widetilde{\psi}\right)+\phi(\widetilde{v})-\phi(\widetilde{\psi})-\left(\theta(\widetilde{\psi},\widetilde{v}),\widetilde{v}-\widetilde{\psi}\right)-\left(f,\left(\widetilde{v}-\widetilde{\psi}\right)\right)\right),\widetilde{p}\right\rangle$$

Carry out a preliminary series of reforms to simplify the original formulation of the problem. Introduce the notation

$$\varphi(t,\bar{z})\cdot\lambda(\psi)=\Phi(\psi),\quad \varphi(t,\bar{z})\cdot\lambda(v)=\Phi(v)$$

and

$$\phi(\psi)=\int_{\Gamma}\Phi(\psi)\,d\Gamma\,,\quad \phi(v)=\int_{\Gamma}\Phi(v)\,d\Gamma\,.$$

In addition, introduce an additional unknown function $\theta(\psi,v)$, the structure corresponding to the functionals $j(\cdot)$, such that

$$(\theta(\psi,v),v-\psi)\geq 0 \quad \forall v\in K .$$

Taking into account the executed transformations introduce the relations (1), (2) in the form

$$\psi\in K:$$

$$\left(m(\bar{z})\frac{\partial\psi}{\partial t},v-\psi\right)+(B(\gamma),v-\psi)+\phi(v)-\phi(\psi)-$$

$$-(\theta(\psi,v),v-\psi)=(f,v-\psi) \quad \forall v\in K . \tag{5}$$

$$\psi(0,\bar{z})=\psi_0(\bar{z}), \tag{6}$$

To solve the problem of finding a state function $\psi(t,\bar{z})$, use an optimization procedure of the Pontryagin maximum principle [10], for which choose the following performance criterion

$$J=min\int_{\Gamma}\int_{0}^{T}|v-\psi|\,dtd\Gamma . \tag{7}$$

The physical meaning of this criterion follows from the next. The trial function $v(t,\bar{z})$ is some approximation of the unknown function $\psi(t,\bar{z})$, reflecting only the essence of physics in the specific process. Therefore, the adequacy of physical processes caused by the action of functions $v(t,\bar{z})$ and $\psi(t,\bar{z})$, is provided up to the accuracy within the difference between these functions. In this case, the integral difference between the trial $v(t,\bar{z})$ and the unknown $\psi(t,\bar{z})$ functions can be regarded as a quantitative measure or a penalty for the deviation of the actual flow of the process from its true value.

Obtain the necessary optimality conditions of the problems (5) (6), (7).

According to [6], introduce a new coordinate

$$\frac{\partial^2\sigma}{\partial t\,\partial z}=|v-\psi|^2\Big|_{z\in\Gamma} . \tag{8}$$

Thus, the original problem will be considered in $(n+1)$-dimensional space with the equation of dynamics

$$\widetilde{\psi}\in K:$$

$$\left(m(\bar{z})\frac{\partial\widetilde{\psi}}{\partial t},\widetilde{v}-\widetilde{\psi}\right)+(B(\gamma),\widetilde{v}-\widetilde{\psi})+\phi(\widetilde{v})-\phi(\widetilde{\psi})-$$

$$-(\theta(\widetilde{\psi},\widetilde{v}),\widetilde{v}-\widetilde{\psi})=(f,\widetilde{v}-\widetilde{\psi}) \quad \forall\widetilde{v}\in K, \tag{9}$$

where

$$\widetilde{\psi}=(\sigma,\psi_1,\ldots,\psi_n),\ \ \widetilde{v}=(\sigma,v_1,\ldots,v_n)$$

with the initial conditions

$$\widetilde{\psi}(0,\bar{z})=[0,\psi_0(\bar{z})].$$

Assume that we have found $\psi(t,\bar{z})$. This condition corresponds to the relation

$$min\int_{\Gamma}\int_{0}^{T}|\widetilde{v}-\widetilde{\psi}|^2\,dtd\Gamma\rightarrow J_{min}=J^{*} .$$

At $t=\tau\,(0\leq\tau\leq T)$ perform a needle-shaped variation with the duration $\varepsilon$. As a result of the variation performed the value of the functional $J$ (7) changes

$$\hat{J}=\int_{\Gamma}\int_{0}^{T}|\widetilde{v}-\widetilde{\psi}|\,dtd\Gamma>J_{min} .$$

Write down the detailed result of the variation

$$\delta\widetilde{v}=\widetilde{v}-\widetilde{\psi}=\varepsilon\{[(B(\gamma)\widetilde{\psi},\widetilde{v}-\widetilde{\psi})+\phi(\widetilde{v})-\phi(\widetilde{\psi})-$$

$$-(\theta(\widetilde{\psi},\widetilde{v}),\widetilde{v}-\widetilde{\psi})-(f,(\widetilde{v}-\widetilde{\psi}))]-(B(\gamma)\widetilde{\psi},\psi)+\phi(\widetilde{\psi})-(\theta(\widetilde{\psi}),\widetilde{\psi})-(f,\widetilde{\psi})\}_{t=\tau} . \tag{10}$$

Express $\widetilde{v}$ through the variation and optimal function of the state

$$\widetilde{v} = \widetilde{\psi} + \delta\widetilde{v}\,. \tag{11}$$

Substituting (11) into (9), obtain

$$\widetilde{\psi} \in K:$$

$$\left( m(\bar{z})\frac{\partial\widetilde{\psi}}{\partial t}, (\widetilde{\psi} + \delta\widetilde{v}) - \widetilde{\psi} \right) = \left( B(\gamma)\widetilde{\psi}, (\widetilde{\psi} + \delta\widetilde{v}) - \widetilde{\psi} \right) + \phi(\widetilde{\psi} + \delta\widetilde{v}) - \phi(\widetilde{\psi}) -$$

$$- \left( \theta(\widetilde{\psi}, (\widetilde{\psi} + \delta\widetilde{v})), (\widetilde{\psi} + \delta\widetilde{v}) - \widetilde{\psi} \right) - \left( f, (\widetilde{\psi} + \delta\widetilde{v}) - \widetilde{\psi} \right) \quad \forall\widetilde{v} \in K\,. \tag{12}$$

For further transformations use the coordinate-wise analog (12)

$$\widetilde{\psi}_i \in K:$$

$$\left( m(\bar{z}_i)\frac{\partial\widetilde{\psi}_i}{\partial t}, (\widetilde{\psi}_i + \delta\widetilde{v}_i) - \widetilde{\psi}_i \right) =$$

$$= \left( B(\gamma)\widetilde{\psi}_i, (\widetilde{\psi}_i + \delta\widetilde{v}_i) - \widetilde{\psi}_i \right) + \phi(\widetilde{\psi}_i + \delta\widetilde{v}_i) - \phi(\widetilde{\psi}_i) -$$

$$- \left( \theta(\widetilde{\psi}_i, (\widetilde{\psi}_i + \delta\widetilde{v}_i)), (\widetilde{\psi}_i + \delta\widetilde{v}_i) - \widetilde{\psi}_i \right) - \left( f, (\widetilde{\psi}_i + \delta\widetilde{v}_i) - \widetilde{\psi}_i \right) \quad \forall\widetilde{v}_i \in K;$$

$$i = 0,1,\dots,n\,. \tag{13}$$

Expand (13) in Taylor series and restrict the consideration with the quantities of 1-th order of infinitesimality

$$m(z_i)\left( \frac{\partial\widetilde{\psi}_i}{\partial t} + \frac{\partial\widetilde{v}_i}{\partial t} \right) =$$

$$= \left( B(\gamma)\widetilde{\psi}_i, \widetilde{\psi}_i \right) + \phi(\widetilde{\psi}_i) - (f, \widetilde{\psi}_i) +$$

$$+ \sum_{i=0}^{n} \frac{\partial\left[ \left( B(\gamma)\widetilde{\psi}_i, \widetilde{\psi}_i \right)_i + \phi(\widetilde{\psi}_i) - (f, \widetilde{\psi}_i) \right]}{\partial\,\widetilde{v}_i}\delta\widetilde{v}_i\,,; \quad i = 0,1,\dots,n\,. \tag{14}$$

From (14) it follows that

$$m(z_i)\frac{\partial\widetilde{v}_i}{\partial t} = \sum_{i=0}^{n} \frac{\partial\left[ \left( B(\gamma)\widetilde{\psi}_i, \widetilde{\psi}_i \right) + \phi(\widetilde{\psi}_i) - (f, \widetilde{\psi}_i) \right]}{\partial\,\widetilde{v}_i}\delta\widetilde{v}_i\,; \quad i = 0,1,\dots,n\,. \tag{15}$$

Now turn to $t = T$. Define a variation of the functional at $t = T$

$$\delta J_{t=T} = \hat{J} - J_{min} > 0 \ \text{ or } \ -\delta J_{t=T} = -\delta\sigma_{\text{Њ=Ѓ}} \le 0\,.$$

Introduce the variable $\widetilde{p}(t,\bar{z})$ so that when $t = T$ this condition is satisfied

$$-\delta J_{t=T} = -\delta\sigma(T) = \left\langle \delta\widetilde{v}, \widetilde{p} \right\rangle_{t=T}\,. \tag{16}$$

Coordinate wise analog (16) is as follows

$$-\delta J_{t=T} = -\delta\sigma(T) = \left\langle \delta\widetilde{v}_i, \widetilde{p}_i \right\rangle_{t=T}; \quad i = 0,1,\dots,n\,.$$

Since $\delta\sigma(T) > 0$, in order to satisfy this relation there should take place:

$$p^0(T,\bar{z}_i) = -1;\ p_j(T,\bar{z}) = 0,$$

where $i = 0,1,\dots,n;\ j = 1,\dots,n$.

Thus, if the optimal solution is not found, then $-\delta J < 0$, and for the optimal solution $-\delta J = 0$ is valid, since the variation of functional must be zero for the optimal solution.

Associate a variable $\widetilde{p}(t,\bar{z})$ to the dynamic equation of the process observed through trial function $v(t,\bar{z})$. Find a variable $\widetilde{p}(t,\bar{z})$ which satisfies

$$\left\langle \delta\widetilde{v}(t,\bar{z}), \widetilde{p}(t,\bar{z}) \right\rangle = \left\langle \delta\widetilde{v}(T,\bar{z}), \widetilde{p}(T,\bar{z}) \right\rangle_{\tau+\varepsilon\le t\le T} = const\,.$$

Then we have

$$\frac{\partial}{\partial t}\left\langle \delta\widetilde{v}(t,\bar{z}), \widetilde{p}(t,\bar{z}) \right\rangle = \left\langle \frac{\partial\,\delta\widetilde{v}(t,\bar{z})}{\partial t}, \widetilde{p}(t,\bar{z}) \right\rangle + \left\langle \frac{\partial\, v\widetilde{p}(t,\bar{z})}{\partial t}, \delta\widetilde{v}(t,\bar{z}) \right\rangle_{\tau+\varepsilon\le t\le T} = 0\,. \tag{17}$$

Coordinatewise analog (17) is

$$\sum_{i=0}^{n}\frac{\partial\,\delta\tilde{v}_i(t,\bar{z})}{\partial t},\tilde{p}_i(t,\bar{z})+\sum_{i=0}^{n}\delta\tilde{v}_i(t,\bar{z})\frac{\partial\,\tilde{p}_i\tilde{v}_i(t,\bar{z})}{\partial t}=0;\;\; i=0,1,\ldots,n. \tag{18}$$

Substitute in (18) the value of the derivative $\frac{\partial\,\delta\tilde{v}(t,\bar{z})}{\partial t}$ from (15)

$$m(z_i)\sum_{i=0}^{n}\tilde{p}_i\times\sum_{i=0}^{n}\frac{\partial\left[\left(B(\gamma)\tilde{\psi}_i,\tilde{\psi}_i\right)+\phi(\tilde{\psi}_i)-\left(f,\tilde{\psi}_i\right)\right]}{\partial\tilde{v}_i}\delta\tilde{v}_i+$$

$$+\sum_{i=0}^{n}\delta\tilde{v}_i\frac{\partial\,\tilde{p}}{\partial t}=0;\;\; i=0,1,\ldots,n. \tag{19}$$

Change the order of summation in (19)

$$m(z_i)\sum_{i=0}^{n}\delta\tilde{v}_i+\left[\sum_{i=0}^{n}\tilde{p}_i\frac{\partial\left[\left(B(\gamma)\tilde{\psi}_i,\tilde{\psi}_i\right)+\phi(\tilde{\psi}_i)-\left(f,\tilde{\psi}_i\right)\right]}{\partial\tilde{v}_i}+\frac{\partial\,\tilde{p}_i}{\partial t}\right]=0;$$

$$i=0,1,\ldots,n.$$

Finally get

$$\frac{\partial\,\tilde{p}_i}{\partial t}=-\sum_{i=0}^{n}\frac{\partial\left[\left(B(\gamma)\tilde{\psi}_i,\tilde{\psi}_i\right)+\phi(\tilde{\psi}_i)-\left(f,\tilde{\psi}_i\right)\right]}{\partial\tilde{v}_i}\tilde{p}_i;\;\; i=0,1,\ldots,n.$$

Note that this equation is the dual of (5), and the variable $\tilde{p}(t,\bar{z})$ is expressed through the function of phase.

Again turn to the variation of functional (7) at $t=T$

$$-\delta J_{t=T}=\left\langle\delta\tilde{v}(t,\bar{z}),\tilde{p}(t,\bar{z})\right\rangle_{t=T}=0.$$

Replace the variation $\delta\tilde{v}$ with the value of (10), reduce by $\varepsilon$ and, since $\tau$ can be arbitrary, obtain

$$\left\langle\left(\left(B(\gamma)\tilde{\psi},\tilde{v}-\tilde{\psi}\right)+\phi(\tilde{v})-\phi(\tilde{\psi})-\left(\theta(\tilde{\psi},\tilde{v}),\tilde{v}-\tilde{\psi}\right)-\left(f,(\tilde{v}-\tilde{\psi})\right)\right),\tilde{p}\right\rangle_{t=\tau}-$$

$$-\left\langle\left(\left(B(\gamma)\tilde{\psi},\tilde{\psi}\right)+\phi(\tilde{\psi})-\left(f,\tilde{\psi}\right)\right),\tilde{p}\right\rangle_{t=T}=0. \tag{20}$$

From (20) it follows that the second summand in it corresponds to the optimal solution of the variational inequality (5). In the case when the optimal solution $\psi(t,\bar{z})$ is found, variation of functional $J$ will be zero, i.e. $\delta J=0$. Given this, the first summand in (20), defined by the Hamiltonian function

$$\tilde{H}=\left\langle\left(\left(B(\gamma)\tilde{\psi},\tilde{v}-\tilde{\psi}\right)+\phi(\tilde{v})-\phi(\tilde{\psi})-\left(\theta(\tilde{\psi},\tilde{v}),\tilde{v}-\tilde{\psi}\right)-\left(f,(\tilde{v}-\tilde{\psi})\right)\right),\tilde{p}\right\rangle, \tag{21}$$

should take the maximum value. Thus, the above statement is proven. Let's show the possibility of determining the maximum value of Hamiltonian function.

Coordinatewise analog (21) is defined by

$$\tilde{H}=\left\langle\left(\left(B(\gamma)\tilde{\psi}_i,\tilde{v}_i-\tilde{\psi}_i\right)+\phi(\tilde{v}_i)-\phi(\tilde{\psi}_i)-\left(\theta(\tilde{\psi}_i,\tilde{v}_i),\tilde{v}_i-\tilde{\psi}_i\right)-\left(f,(\tilde{v}_i-\tilde{\psi}_i)\right)\right),\tilde{p}_i\right\rangle;$$

$$i=0,1,\ldots,n. \tag{22}$$

To maximize the value of the function $\tilde{H}$, it's necessary to set all the partial derivatives of this function to zero by a testing variable $v(t,\bar{z})$, that taking into account (22) gives the system of equations

$$\frac{\partial\tilde{H}}{\partial v_i}=0;\;\; i=0,1,\ldots,n. \tag{23}$$

Coordinate wise analog (22) contains $(n+1)$ of $v_i$ functions, $(n+1)$ of $\theta_i$ functions and $(n+1)$ of $p_i$ functions. Since the equations (23) are only $(n+1)$, and the unknown are $(3n+3)$, then the system (23) cannot be solved. To solve (23) define also the partial derivatives

$$\frac{\partial \widetilde{H}}{\partial \theta_i}=\widetilde{p}_i;\quad i=0,1,\ldots,n. \tag{24}$$

$$\frac{\partial \widetilde{H}}{\partial p_i}=\left[m(\bar{z}_i)\frac{\partial \widetilde{\psi}_i}{\partial t},\widetilde{v}_i-\widetilde{\psi}_i\right],\quad i=0,1,\ldots,n. \tag{25}$$

In this case, the solution of (23) can be obtained.

As a result of the reasoning done, the scheme of the algorithm for solving variational inequality (5) using the maximum principle can be represented as follows:

1. The dynamic equation (9), subject to the additional coordinate $\sigma$ is written down.
2. An auxiliary function (Hamilton) $\widetilde{H}$ in accordance with the expression (22) is compiled.
3. A test function $v(t,\bar{z})$ that delivers maximum $\widetilde{H}$ functions in accordance with the expression (23) is determined. For the redefinition of the independent variables $\theta$ and $p$ the system (23) is supplemented with equations (24) and (25).
4. The unknown variable $\psi(t,\bar{z})$ is determined by the test variable $v(t,\bar{z})$, which gives the maximum value of function $\widetilde{H}$.

**2. Method of parametrical identification of abnormal diffusive processes.** At statement of an inductive task (1) – (4), the method focused on numerical machine realization can be offered parametrical identification of MM of a look. The essence of a method consists in the following

It agrees [11], to MM (1) – (4) (in increments) it is possible to present in a look

$$-\frac{m\,\partial(\Delta\psi)}{\partial t}-\int_{\Omega}\sum_{i=1}^{n}\left[B(\gamma)\frac{\partial^2(\Delta\psi)}{\partial z_i^2}|\Delta v|\right]dz\geq\sum_{j=1}^{k}\zeta_j(z)f_j;\quad\forall\psi,v\in K\quad, \tag{26}$$

$$\Delta\psi(0,z)=\Delta\psi_0(z), \tag{27}$$

where $\psi=\psi(t,z)$ — sought function; $v=v(t,z)$ - trial function; $K$ - a lot, of that is defined functions $\psi=\psi(t,z)$ and $v=v(t,z)$; $f$ - exciting function; $k$ - number of exciting functions; $\zeta(z)$- Dirac's function; $m=m(\cdot)$ and $B=B(\cdot)$ - identified parameters.

As criterion of quality of the solution of a problem of identification we will accept functionality of a look

$$J[m(\cdot),B(\cdot)]=\sum_{j=1}^{k}\left\{\int_{T_j}\left[\psi'(t,z,m,B)-F_j^{\psi}(t)\right]^2dt\right\}, \tag{28}$$

where $\psi'(t,z,m,B)$ - exact values sought functions; $F_j^{\psi}(t)$ - measured values of the sought function; $T$ - time of measurements.

Let's show that the accepted criterion of quality will be differentiable in any point of spatial area $\bar{z}\in\Omega$ (including and its border $\Gamma$), i.e. an increment (28) equal

$$\Delta J=J\left[\left(m+h^m\right),\left(B+h^B\right)\right]-J(m,B)$$

represent able in a look

$$\Delta J=\int_{\Omega}\left\{\left[J'(m,B)h^m\right]dz+\left[J'(m,B)h^B\right]dz\right\}+\left[O\left(\left\|h^m\right\|_{L^2}\right)+O\left(\left\|h^B\right\|_{L^2}\right)\right], \tag{29}$$

where $J'(m,B)$ — some function from $L^2(\Omega)$; $O\left(\left\|h^m\right\|_{L^2}\right)$ and $O\left(\left\|h^B\right\|_{L^2}\right)$ - residual members such, that $\lim\limits_{\alpha^m \to 0} \left[ O(\alpha^m)(\alpha^m)^{-1} \right] = 0$, $\lim\limits_{\alpha^m \to 0} \left[ O(\alpha^m)(\alpha^m)^{-1} \right] = 0$.

Let’s write down formally a functionality increment

$$\Delta J = \sum_{j=1}^{k} \left\{ \int_{\Omega} \left\{ \left[ \psi(t, z_j, v, B) + \Delta\psi(t, z_j) - F_j^{\psi}(t) \right]^2 - \left[ \psi(t, z, m, B) - F_j^{\psi}(t) \right] \right\} dz \right\} =$$

$$= \sum_{j=1}^{k} \left\{ \int_{\Omega} \left\{ \left\{ \left[ \psi(t, z, m, B) - F_j^{\psi}(t) \right] + \Delta\psi(t, z_j) \right\}^2 - \left[ \psi(t, z_j, v, B) - F_j^{\psi}(t) \right] \right\} dz \right\} =$$

$$= \sum_{j}^{k} \left\{ \left\{ \int_{\Omega} 2 \left[ \psi(t, z, m, B) - F_j^{\psi}(t) \right] \Delta\psi(t, z) dz + \int_{\Omega} \Delta\psi^2(t, z) \right\} . \qquad (30)$$

Let’s transform this expression to a look (29). For this purpose we will enter into consideration of function $p_{\psi}^{*}(t, z) \equiv p_{\psi}^{*}(t, z, m, B)$ as the solution of the following regional task

$$-\frac{m \partial p_{\psi}^{*}}{\partial t}(v - \psi) - \int_{\Omega} \sum_{i=1}^{n} \left[ B(\gamma) \frac{\partial^2 p_{\psi}^{*}}{\partial z_i^2} |\Delta v| \right] dz \geq \sum_{j=1}^{k} \zeta_j(z) f_j ; \quad \forall \psi, v \in K \qquad , \qquad (31)$$

$$p_{\psi}^{*}\Big|_{t = t_k} = 2\left[ \psi(t_k, z, m, B) - F_j^{\psi}(t) \right], p_{\psi}^{*}\Big|_{t = t_k}, \quad \forall z \in \Omega . \qquad (32)$$

The first integral in the first composed in the right part of equality (30) taking into account (26), (27), (31), (32) it will be transformed so

$$I = 2\left[ \psi(t, z, m, B) - F_j^{\psi}(t) \right] \Delta\psi(t, z) dz =$$

$$= \int_{\Omega} p_{\psi}^{*}(t_k, z_j) \Delta\psi(t_k, z_j) = \int_{\Omega} \left[ \int_{0}^{t_k} \frac{\partial}{\partial t} (p_{\psi}^{*}, \Delta\psi) dt \right] dz =$$

$$= \int_{\Omega} \int_{0}^{t_k} \left[ \frac{\partial p_{\psi}^{*}}{\partial \psi} \Delta\psi + p_{\psi}^{*} \frac{\partial(\Delta\psi)}{\partial t} \right] dt\, dz = \int_{\Omega} \int_{0}^{t_k} \left\{ \frac{1}{m(\cdot)} \left\{ \sum_{i=1}^{n} \left[ B(\cdot) \frac{\partial^2 p_{\psi}^{*}}{\partial z_i^2} |v| \right] \right\} \Delta\psi + \right.$$

$$\left. + p_{\psi}^{*} \left\{ \sum_{i=1}^{nk} \left[ B(\cdot) \frac{\partial^2 p_{\psi}^{*}}{\partial z_i^2} |\Delta v| \right] \right\} \right\} dt\, dz .$$

Integrating the last expression in spatial area, we will receive the following result

$$I_{\psi}' = \int_{0}^{t_k} \left\{ \frac{1}{m(\cdot)} \left\{ \sum_{i=1}^{n} \left[ B(\cdot) \frac{\partial^2 p_{\psi}^{*}}{\partial z_i^2} |v| \right] \right\} \Delta\psi + p_{\psi}^{*} \left\{ \sum_{i=1}^{n} \left[ B(\cdot) \frac{\partial^2 p_{\psi}^{*}}{\partial z_i^2} |\Delta v| \right] \right\} \right\} \Bigg|_{\Omega} dt =$$

$$= \int_{0}^{t_r} \frac{1}{m(\cdot)} \left[ \left( \sum_{i=1}^{n} B(\cdot) p_{\psi}^{*} |v| \right) \right] \Delta\psi\, dt . \qquad (33)$$

Here, and further, designation $(\cdot)$ determines as the linear (from space), and non-linear (from required function) parameter. The second integrals in composed in the right part (30) the members of the look $\left[ O\left( \|h\|_{L_2}^{\psi} \right) \right]$, presented in (29) and written down for spatial problem definition define. Let’s have in this case

$$\Delta J = \int_{\Omega} \frac{1}{m(\cdot)} \left[ \left( \sum_{i=1}^{n} B(\cdot) p_{\psi}^{*} |v| \right) \right] h^{\psi} dz + O\left( \|h\|_{L_2}^{\psi} \right), \qquad (34)$$

and, the step $h^\psi$ determines cooperative value by steps $h^m$ and $h^B$. As a result we will receive that the increment of functionality (28) is represented in the form of expression

$$\Delta J = \int_\Omega \frac{1}{m(\cdot)}\left[\left(\sum_{i=1}^{n} B(\cdot)p_\psi^*|v|\right)\right]h^\psi dz + O\left(\|h\|_{L_2}^{\psi}\right).$$

Thus, required representation (29) for functionality (28) is received, and the gradient of this functionality looks like

$$J'[m(\cdot),B(\cdot)] \equiv \frac{1}{m(\cdot)}\left[\left(\sum_{i=1}^{n} B(\cdot)p_\psi^*|v|\right)\right], \quad \forall z \in \Omega, t \in [0,t_k]. \tag{35}$$

Further, having a gradient (35) and using procedure of a method of a projection of the gradient [11], defined by ratios

$$Q = \{q(t): q(t) \in L_2[0,t_k], a \le q(t) \le b, \forall t \in [0,t_k]\}$$

$$Pr_q[q(t)] = \begin{cases} q(t), & a \le q(t) \le b, \\ a, & q(t) < a, \\ b, & q(t) > b. \end{cases}$$

For identified functions $m(\cdot)$ and $B(\cdot)$ also we will receive final ratios on an offered method of parametrical identification

$$m_{u+1}(\cdot) = \begin{cases} m_r - \dfrac{\alpha_m}{m(\cdot)}\left[\left(\sum\limits_{i=1}^{n} B(\cdot)p_\psi^*|v|\right)\right]; \\ m_{min} \le m_r - \dfrac{\alpha_m}{m(\cdot)}\left[\left(\sum\limits_{i=1}^{n} B(\cdot)p_\psi^*|v|\right)\right] \le \\ \le m_{max}; \\ m_{min},\, m_r - \dfrac{\alpha_m}{m(\cdot)}\left[\left(\sum\limits_{i=1}^{n} B(\cdot)p_\psi^*|v|\right)\right] < m_{min}; \\ m_{max},\, m_r - \dfrac{\alpha_m}{m(\cdot)}\left[\left(\sum\limits_{i=1}^{n} B(\cdot)p_\psi^*|v|\right)\right] > m_{max}, \end{cases}$$

$$B_{u+1}(\cdot) = \begin{cases} B_r - \dfrac{\alpha_B}{m(\cdot)}\left[\left(\sum\limits_{i=1}^{n} B(\cdot)p_\psi^*|v|\right)\right]; \\ B_{min} \le B_r - \dfrac{\alpha_B}{m(\cdot)}\left[\left(\sum\limits_{i=1}^{n} B(\cdot)p_\psi^*|v|\right)\right] \le \\ \le B_{max}; \\ B_{min},\, B_r - \dfrac{\alpha_B}{m(\cdot)}\left[\left(\sum\limits_{i=1}^{n} B(\cdot)p_\psi^*|v|\right)\right] < B_{min}; \\ B_{max},\, B_r - \dfrac{\alpha_B}{m(\cdot)}\left[\left(\sum\limits_{i=1}^{n} B(\cdot)p_\psi^*|v|\right)\right] > B_{max}, \end{cases}$$

where $\alpha_m$ and $\alpha_B$ - method parameters, defined by practical consideration, $r$ — step of the numerical decision.

**Conclusion.** The conducted numerical researches showed that the offered methods of mathematical model operation and parametrical identification of the abnormal diffusion processes, based on iterative procedures of optimization possess good convergence (the decision is reached no more, than for 8 – 10 iterations) at accuracy of the decision 0,2% are not lower.

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***Для класу аномальних дифузійних процесів, математичні моделі яких формалізовано у вигляді варіаційних нерівностей у частинних похідних, запропоновано метод математичного моделювання на основі процедури оптимізації. Метод розглянуто відносно узагальненої математичної моделі досліджуваного класу аномальних дифузійних процесів. Що дало змогу забезпечити принцип уніфікації та типізації у застосуванні даного методу, а також коректність використання узагальненої математичної моделі в прикладних задачах математичного моделювання відомих промислових та практично важливих природних випадків аномальних дифузійних процесів. При цьому задача реалізації математичних моделей аномальних дифузійних процесів на основі запропонованого методу зводиться до пошуку максимуму функції Гамільтона, визначеної у просторі станів процесів, які розглядаються. Також запропоновано метод параметричної ідентифікації математичних моделей аномальних дифузійних процесів у постановці задачі оптимального управління. Метод зводиться до використання оптимізаційної процедури методу проекції градієнта. Доведено можливість розв’язання задачі параметричної ідентифікації у випадках як лінійної, так і нелінійної математичних моделей аномальних дифузійних процесів. Причому, нелінійна постановка задачі параметричної ідентифікації не призводить до ускладнень обчислювальної реалізації, оскільки розв’язок ґрунтується лише на віднаходженні проєкції градієнта функції стану аномального дифузійного процесу. Запропоновані методи викладено із строгим дотриманням положень функціонального аналізу, що забезпечує їхню коректність та адекватність при розв’язуванні широкого кола прикладних задач.***

***Ключові слова: аномальний дифузійний процес, математична модель, варіація, варіаційна нерівність, оптимізація, принцип уніфікації та типізації, градієнт, параметрична ідентифікація.***